

Generalized asymptotic structure of the ultrarelativistic Schwarzschild black hole

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Abstract

We discuss the asymptotic structure of the ultrarelativistic Schwarzschild black hole. An explicit construction for a conformal boundary both at spatial and null infinity is given together with the corresponding expressions for the ADM and Bondi four-momenta.

TUW-99-28
UWthPh-1999-78

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[†]supported by the Austrian science foundation: project P13007-PHY

Introduction

In a previous short comment the authors [1] argued that it is possible to assign ADM as well as Bondi four-momentum to the geometry of an ultrarelativistic Schwarzschild black hole, which is described by the so-called Aichelburg-Sexl metric (AS). This result is in agreement with the physically motivated expectations: the Bondi momentum is lightlike and equal to the ADM momentum up to the instant of retarded time when both, particle and radiation, escape to infinity and drops to zero thereafter, leaving flat space behind. The naive calculation made use of the standard limiting procedure to spatial and null infinity of global quantities defined with respect to manifestly asymptotic flat coordinates. Although it lead to the intuitively correct results the asymptotic structure of the AS metric was not discussed.

The goal of the present work is somewhat more ambitious and aims at the construction of asymptopia, i^0 as well as \mathcal{I} , for the AS-metric. Due to the distributional nature of the geometry, which belongs to the class of impulsive gravitational waves, the usual techniques do not immediately apply. Therefore, to disentangle geometrical and distributional effects we also consider the analogous electromagnetic situation in Minkowski space. The electromagnetic field of an ultrarelativistic charge has a distributional structure similar to that of the impulsive gravitational wave. In the first part of section two (em-pulse), we show that the integrals of the field at spatial and null infinity encode the correct total charge.

To exhibit our approach we first consider in section one the unboosted situation and construct the asymptotic Coulomb and Schwarzschild fields. This leads, of course, to well-known results but allows us to introduce notation and the techniques we wish to apply: For spatial infinity we essentially follow the Ashtekar-Hansen [2, 3] approach whereas for null infinity we make use of the Penrose-like definitions, however in terms of tensor quantities. For the construction of the conformal boundary of the Schwarzschild field we use the flat part of the Kerr-Schild decomposition. This will serve as a starting point for the extension to the impulsive situation.

The main part of this paper can be found in second part of section two (AS-geometry) where we apply the formalism to the boosted Schwarzschild metric. These calculations not only show that it makes sense to construct a conformal boundary for this spacetime, although in a somewhat generalized form, but also confirm our previous results of total four-momentum.

Section zero discusses the standard form of the AS-metric and gives a brief introduction to the concepts of direction-dependent limits and its distributional extension.

We would like to emphasize that nowhere we make use of non-linear

techniques for generalized functions such as Colombeau theory. All quantities are well defined in the setting of classical distributional theory.

0) *Preliminaries*

It is well known that the metric

$$\begin{aligned} ds^2 &= -dt^2 + dz^2 + dx^2 + dy^2 + f(dt - dz)^2, \\ f &= -8\mu\delta(t - z) \log \sqrt{x^2 + y^2} \end{aligned} \quad (1)$$

can be obtained from the Schwarzschild geometry by applying a singular boost (or light-like contraction) i.e. taking the limit $\gamma \rightarrow \infty$ and at the same time the mass $m \rightarrow 0$ such that $m\gamma = \mu = \text{const.}$

Geometrically this metric is defined by

1. the existence of a covariantly constant vector field,
2. satisfying Einsteins vacuum equations (except on a null line),
3. a curvature tensor which is completely concentrated on a null hyperplane and that
4. its symmetry group acts on the null hyperplanes and has the structure $\mathbb{R} \times ISO(2)$

The first two conditions imply that the metric belongs to the class of pp-waves which in turn guarantees the existence of adapted coordinates such that the metric takes the above form with the a general profile $f = f(t - z, x, y)$ being only restricted by $(\partial_x^2 + \partial_y^2)f = 0$.

So far the coordinates are only determined up to a (still infinite-dimensional) subgroup of diffeomorphism group, namely the so-called “normal form preserving” transformations. In a classical work [4] pp-waves have been characterized by listing all possible symmetry groups together with a corresponding canonical representative. This classification has been extended in [5, 6] to encompass also distributional wave-profiles. It follows that for impulsive pp-waves the normal form preserving diffeomorphisms are reduced to a seven-dimensional Lie group. The last condition then determines the form of the metric (1) up to a rescaling of the null vector field $p^a = \partial_t^a + \partial_z^a \rightarrow \lambda p^a$, which in turn rescales μ . If however one insists that the metric is obtained from Schwarzschild in the ultrarelativistic limit then μ is fixed by the value of m .

Since the AS-geometry is distributional in nature and we are interested in its asymptotic structure, we have to deal with asymptotic limits of distributional quantities. Therefore we start with an informal account of direction-dependent limits and by means of a simple example give the extension of this

notion to generalized functions (distributions). Let us begin with the (Euclideanized) notion of direction dependent limits. Direction dependent limits are taken with respect to a definite point p of M and its corresponding tangent space $T_p M$. It is therefore useful to first state our coordinate-conventions: we consider a chart containing $p \in U \subset M$ such that the local coordinates (x^1, \dots, x^n) of p coincide with the origin. Since $T_p M$ consists of all (equivalence classes) of curves through p , it can conveniently be represented by $x^i = \lambda X^i$, where X^i denote the components of the tangent vector with respect to the coordinate basis ∂_i , thereby emphasizing their role as coordinates of $T_p M$, when the latter is considered as a manifold in its own right. Due to the linear structure of $T_p M$ we may identify ∂_{X^i} , with ∂_i . The direction dependent limit of a scalar function f is then defined by

$$\lim_{\lambda \rightarrow 0} f(x(\lambda)) =: F(X) \quad X = \dot{x}(0),$$

provided it only depends on the tangent vector. The limit (if it exists) does actually only depend on the projective class of X . If the manifold is equipped with a (Riemannian) metric it may be used to interpret F as a function on the (unit) S^{n-1} in $T_p M$ by requiring $X^2 = 1$. This approach has been successfully applied (in the Lorentzian context) to investigate the asymptotic structure of spacetime [2, 3].

In order to generalize this notion to allow for generalized function consider the following example: \mathbb{R}^2 equipped with the standard Euclidean metric. The asymptotic limit of a function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is calculated by evaluating

$$\lim_{\rho \rightarrow \infty} F(\rho \cos \phi, \rho \sin \phi) =: f(\phi).$$

This limit (if it exists) will in general depend on the choice of direction along which we approach infinity. In a more geometrical picture we may think of \mathbb{R}^2 as being compactified to S^2 and interpret this limit as a function f on the unit- S^1 in the tangent space at the “north pole” of S^2 representing the infinitely distant point of \mathbb{R}^2 . Making use of test forms $\tilde{\varphi} \in \Omega_0^\infty(S^1)^\dagger$ defined with respect to this S^1 , the limit gives rise to a regular functional f via

$$\lim_{\rho \rightarrow \infty} \int F(\rho \cos \phi, \rho \sin \phi) \varphi(\phi) d\phi =: (f, \tilde{\varphi}).$$

In order to see how this definition extends to singular functionals let us raise the question what the asymptotic limit of $\delta(y)$ – a genuine distribution on \mathbb{R}^2

[†]Here and in the following distributions will be regarded as linear functionals acting on C^∞ test n -forms of compact support rather than test functions. This point of view is the natural one for generalized scalar functions, since the corresponding regular functionals are generated via integration of a scalar function against an n -form.

in global Cartesian (x, y) coordinates – is. At first glance it may seem that this limit does not exist since $\delta(y)$ has the same (infinite) value at its support $y = 0$. However, taking a closer look and using the above prescription for regular functionals we find

$$\lim_{\rho \rightarrow \infty} (\delta(y), \tilde{\varphi}) = \lim_{\rho \rightarrow \infty} (\delta(\rho \sin \phi), \tilde{\varphi}) = \lim_{\rho \rightarrow \infty} \frac{1}{\rho} (\varphi(0) + \varphi(\pi)) = 0.$$

This result is not as surprising as it might look in the first place. It exhibits the negative homogeneity of the delta function as the reason for the vanishing of the limiting distribution. It is precisely this behavior which will turn out to be crucial for the existence of the asymptotic quantities in the following sections.

1) *Coulomb charge and Schwarzschild four-momentum*

Coulomb

Consider the Coulomb potential and its field strength in Minkowski space M

$$\begin{aligned} A &= -\frac{e}{r} dt, \\ F &= dA = \frac{e}{r^2} dr \wedge dt & \begin{aligned} u &= t - r \\ v &= t + r \end{aligned} \\ &= \frac{2e}{(v - u)^2} dv \wedge du. \end{aligned}$$

- i^0 and ADM-charge: In order to evaluate the (total) ADM charge let us switch to a regular coordinate system at (spatial) infinity

$$\begin{aligned} \bar{v} &= -1/u & d\bar{v} &= (1/u^2) du \\ \bar{u} &= -1/v & d\bar{u} &= (1/v^2) dv \end{aligned}$$

With respect to these coordinates F becomes

$$F = \frac{2e}{(-1/\bar{u} + 1/\bar{v})^2} \frac{1}{\bar{u}^2} \frac{1}{\bar{v}^2} d\bar{u} \wedge d\bar{v} = \frac{2e}{(\bar{v} - \bar{u})^2} d\bar{u} \wedge d\bar{v},$$

and the metric

$$ds^2 = -dudv + \frac{(v - u)^2}{4} d\Omega^2 = \frac{1}{\bar{u}^2 \bar{v}^2} (-d\bar{u} d\bar{v} + \frac{(\bar{v} - \bar{u})^2}{4} d\Omega^2) \quad \Omega = -\bar{u}\bar{v}.$$

Following [2, 3] we take the direction dependent limit of F using $\bar{u} = \lambda U, \bar{v} = \lambda V$ and $\lambda \rightarrow 0$ and find

$$\lim_{\lambda \rightarrow 0}(\Omega F) = (-\lambda^2 UV) \frac{2e}{\lambda^2(V-U)^2} dU \wedge dV = (-UV) \frac{2e}{(V-U)^2} dU \wedge dV,$$

which is a tensor-field in $T_{i^0} \hat{M}$, where \hat{M} refers to the standard conformal compactification of M . Physically only the (spatial) projective information is relevant. This information is most conveniently encoded in the hyperboloid $\mathcal{D} : UV = -1$ and the field-strength tensor at \mathcal{D}

$$\mathbb{F} := \lim_{\lambda \rightarrow 0}(\Omega F)|_{\mathcal{D}} = \frac{2e}{(V+1/V)^2} dU \wedge dV,$$

which is a section of the pull-back bundle of the tangent bundle of $T_{i^0} \hat{M}$ to \mathcal{D} . Using the normal $\eta^a = U \partial_U^a + V \partial_V^a$ of \mathcal{D} the flux 2-form $\frac{1}{2} {}^4 \epsilon_{abcd} \mathbb{F}^{ab}$ may be decomposed

$$\frac{1}{2} {}^4 \epsilon_{abcd} \mathbb{F}^{ab} = -(\eta_a \mathbb{F}^{ab}) {}^3 \epsilon_{bcd} - \eta_{[c} {}^3 \epsilon_{abd]} \mathbb{F}^{ab}, \quad {}^4 \epsilon_{abcd} = -\eta_{[a} {}^3 \epsilon_{bcd]}.$$

Integration over a 2-sphere section of \mathcal{D} gives the total or ‘‘ADM’’ charge

$$Q^{ADM} = -\frac{1}{4\pi} \int_{S^2} \frac{1}{2} {}^4 \epsilon_{abcd} \mathbb{F}^{ab} = \frac{1}{4\pi} \int_{S^2} {}^3 \epsilon_{abc} E^a = e$$

$$E := \eta \rfloor \mathbb{F} = -\frac{4e}{(V+1/V)^2} \frac{dV}{V} = -\frac{e}{\text{ch}^2 \tau} d\tau, \quad \{V = e^\tau\}$$

where the second integral expression is the usual Ashtekar expression for the ADM-charge.

• \mathcal{S}^+ and Bondi-charge: As for the ADM-charge we begin with the field strength F and pass to coordinates regular in a neighborhood of \mathcal{S}^+

$$\begin{aligned} F &= \frac{2e}{(v-u)^2} dv \wedge du \quad v = \frac{1}{\bar{v}} \\ &= \frac{2e}{(1-u\bar{v})^2} \frac{1}{v^2} \left(-\frac{d\bar{v}}{\bar{v}^2} \right) \wedge du \\ &= \frac{2e}{(1-u\bar{v})^2} du \wedge d\bar{v}. \end{aligned}$$

In the limit $\bar{v} \rightarrow 0$ this becomes

$$\mathbb{F} := \lim_{\bar{v} \rightarrow 0} F = 2e du \wedge d\bar{v}.$$

Taking the corresponding transformation of the metric into account

$$ds^2 = -dudv + \frac{(v-u)^2}{4}d\Omega^2 = \frac{1}{\bar{v}^2}(dud\bar{v} + \frac{(1-u\bar{v})^2}{4}d\Omega^2) \quad \Omega = \bar{v}$$

allows us to calculate the flux 2-form and the Bondi-charge as its integral over a 2-sphere section of \mathcal{I}^+

$$\begin{aligned} *\mathbb{F} &= *(-4e(e^u \wedge e^{\bar{v}})) = 4e\frac{1}{4}d^2\Omega = ed^2\Omega \\ Q^{Bondi} &= \frac{1}{4\pi} \int_{S^2} *\mathbb{F} = e, \end{aligned}$$

which remains constant since no charge escapes to null-infinity.

Schwarzschild

In the corresponding gravitational situation the Riemann (Weyl) tensor plays the role of the field strength F_{ab} . Using adapted coordinates for the Kerr-Schild decomposition of Schwarzschild

$$g_{ab} = \eta_{ab} + f k_a k_b \quad f = \frac{2m}{r}, \quad k^a = \partial_t^a + \partial_r^a,$$

the mixed form of the Riemann-tensor $R^{ab}{}_{cd}$ (**Riemann**) becomes

$$\begin{aligned} \mathbf{Riemann} &= \frac{f''}{2}(\partial_t \wedge \partial_r)(dt \wedge dr) \\ &\quad + \frac{f'}{2r} \left((\partial_t \wedge \tilde{E}_i)(dt \wedge \tilde{e}^i) + (\partial_r \wedge \tilde{E}_i)(dr \wedge \tilde{e}^i) \right) \\ &\quad + \frac{f}{2r^2}(\tilde{E}_i \wedge \tilde{E}_j)(\tilde{e}^i \wedge \tilde{e}^j) = \left\{ \begin{matrix} u = t - r \\ v = t + r \end{matrix} \right\} \\ &= \frac{f''}{2}(\partial_u \wedge \partial_v)(du \wedge dv) \\ &\quad + \frac{f'}{2r} \left((\partial_u \wedge \tilde{E}_i)(du \wedge \tilde{e}^i) + (\partial_v \wedge \tilde{E}_i)(dv \wedge \tilde{e}^i) \right) \\ &\quad + \frac{f}{2r^2}(\tilde{E}_i \wedge \tilde{E}_j)(\tilde{e}^i \wedge \tilde{e}^j), \end{aligned}$$

where \tilde{E}_i, \tilde{e}^i refer to an orthonormal dyad and co-dyad for the unit two-sphere respectively.

• i^0 and ADM-momentum: Changing coordinates to a patch that is regular at spatial infinity

$$\begin{aligned} \bar{u} &= -1/v & \partial_u &= \bar{u}^2 \partial_{\bar{u}} & du &= 1/\bar{u}^2 d\bar{u} \\ \bar{v} &= -1/u & \partial_v &= \bar{v}^2 \partial_{\bar{v}} & dv &= 1/\bar{v}^2 d\bar{v} \end{aligned}$$

the expression for **Riemann** becomes

$$\begin{aligned} \mathbf{Riemann} = & \frac{f''}{2}(\partial_{\bar{u}} \wedge \partial_{\bar{v}})(d\bar{u} \wedge d\bar{v}) + \\ & + \frac{f'}{2r} \left((\partial_{\bar{u}} \wedge \tilde{E}_i)(d\bar{u} \wedge \tilde{e}^i) + (\partial_{\bar{v}} \wedge \tilde{E}_i)(d\bar{v} \wedge \tilde{e}^i) \right) \\ & + \frac{f}{2r^2}(\tilde{E}_i \wedge \tilde{E}_j)(\tilde{e}^i \wedge \tilde{e}^j). \end{aligned}$$

Taking the direction dependent limit by using $\bar{u} = \lambda U$, $\bar{v} = \lambda V$ and $\lambda \rightarrow 0$ we find

$$\begin{aligned} \lim_{\lambda \rightarrow 0}(\Omega^{-3/2} \mathbf{Riemann}) = & \frac{16M(-UV)^{3/2}}{(V-U)^3}(\partial_U \wedge \partial_V)(dU \wedge dV) \\ & - \frac{8M(-UV)^{3/2}}{(V-U)^3} \left((\partial_U \wedge \tilde{E}_i)(dU \wedge \tilde{e}^i) + (\partial_V \wedge \tilde{E}_i)(dV \wedge \tilde{e}^i) \right) \\ & + \frac{8M(-UV)^{3/2}}{(V-U)^3}(\tilde{E}_i \wedge \tilde{E}_j)(\tilde{e}^i \wedge \tilde{e}^j). \end{aligned}$$

Since, as in the electromagnetic case, only the (spatially) projective information is physically relevant we will use the asymptotic limit of the Riemann tensor at the hyperboloid $\mathcal{D} : UV = -1$

$$\begin{aligned} \mathbb{R}^{ab}{}_{cd} := \lim_{\lambda \rightarrow 0}(\Omega^{-3/2} R^{ab}{}_{cd})|_{\mathcal{D}} = & \frac{16M}{(V+1/V)^3}(\partial_U \wedge \partial_V)^{ab}(dU \wedge dV)_{cd} \\ & - \frac{8M}{(V+1/V)^3} \left((\partial_U \wedge \tilde{E}_i)^{ab}(dU \wedge \tilde{e}^i)_{cd} + (\partial_V \wedge \tilde{E}_i)^{ab}(dV \wedge \tilde{e}^i)_{cd} \right) \\ & + \frac{8M}{(V+1/V)^3}(\tilde{E}_i \wedge \tilde{E}_j)^{ab}(\tilde{e}^i \wedge \tilde{e}^j)_{cd}. \end{aligned}$$

The asymptotic form of the bivector potential is obtained from

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \Omega^{-3/2}(x \wedge \alpha) = & -(\alpha^t - (\alpha \cdot e_r)) \frac{UV^2}{(-UV)^{3/2}} \partial_U \wedge \partial_V \\ & + (\alpha^t + (\alpha \cdot e_r)) \frac{U^2V}{(-UV)^{3/2}} \partial_U \wedge \partial_V + \frac{UV^2}{(-UV)^{3/2}} \frac{2(\alpha \cdot e_\theta)}{V-U} \partial_V \wedge \partial_\theta \\ & + \frac{U^2V}{(-UV)^{3/2}} \frac{2(\alpha \cdot e_\theta)}{(V-U)} \partial_U \wedge \partial_\theta + \frac{UV^2}{(-UV)^{3/2}} \frac{2(\alpha \cdot e_\phi)}{(V-U) \sin \theta} \partial_V \wedge \partial_\phi \\ & + \frac{U^2}{(-UV)^{3/2}} \frac{2(\alpha \cdot e_\phi)}{(V-U) \sin \theta} \partial_U \wedge \partial_\phi \end{aligned}$$

where α denotes a translation with respect to flat part of the Kerr-Schild decomposition. The expression for the bivector potential at \mathcal{D} becomes

$$\begin{aligned} Q := \lim_{\lambda \rightarrow 0} \Omega^{-3/2} (x \wedge \alpha)|_{\mathcal{D}} &= (\alpha^t - (\alpha \cdot e_r)) V \partial_U \wedge \partial_V \\ &+ (\alpha^t + (\alpha \cdot e_r)) \frac{1}{V} \partial_U \wedge \partial_V - \frac{2(\alpha \cdot e_\theta)}{V + 1/V} V \partial_V \wedge \partial_\theta + \frac{2(\alpha \cdot e_\theta)}{V + 1/V} \frac{1}{V} \partial_U \wedge \partial_\theta \\ &- \frac{2(\alpha \cdot e_\phi)}{(V + 1/V) \sin \theta} V \partial_V \wedge \partial_\phi + \frac{2(\alpha \cdot e_\phi)}{(V + 1/V) \sin \theta} \frac{1}{V} \partial_U \wedge \partial_\phi. \end{aligned}$$

Contraction of $\mathbb{R}^{ab}{}_{cd}$ with Q^{ab} yields the gravitational field-strength. Using the normal $\eta^a = U \partial_U^a + V \partial_V^a$ of \mathcal{D} together with the expression for an asymptotic translation

$$\begin{aligned} \lim_{\lambda \rightarrow 0} (\Omega^{-1} \alpha^a)|_{\mathcal{D}} &= (\alpha^t - (\alpha \cdot e_r)) V^2 \partial_V^a + (\alpha^t + (\alpha \cdot e_r)) \frac{1}{V^2} \partial_U^a \\ &+ (\alpha \cdot e_\theta) \frac{2}{V + 1/V} \partial_\theta^a + (\alpha \cdot e_\phi) \frac{2}{V + 1/V} \partial_\phi^a = \\ &= (\alpha^t \operatorname{ch} \tau - (\alpha \cdot e_r) \operatorname{sh} \tau) \partial_\tau^a + \\ &(\alpha \cdot e_\theta) \frac{1}{\operatorname{ch} \tau} \partial_\theta^a + (\alpha \cdot e_\phi) \frac{1}{\operatorname{ch} \tau \sin \theta} \partial_\phi^a \end{aligned}$$

and taking into account that $Q^{ab} = -(\eta \wedge \alpha)^{ab*}$, the corresponding flux 2-form may be decomposed as

$$\frac{1}{4} \epsilon_{abcd} \mathbb{R}^{ab}{}_{mn} Q^{mn} = {}^3\epsilon_{bcd} (\eta_a \mathbb{R}^{ab}{}_{mn} \eta^m) \alpha^n + \eta_{[c} {}^3\epsilon_{d]ab} \mathbb{R}^{ab}{}_{mn} \eta^m \alpha^n.$$

Integration of the flux 2-form over an S^2 -section of \mathcal{D} gives the ADM-momentum

$$P^{ADM} \cdot \alpha = -\frac{1}{8\pi} \int_{S^2} \frac{1}{4} \epsilon_{abcd} (\mathbb{R}^{ab}{}_{mn} Q^{cd}) = -\frac{1}{8\pi} \int_{S^2} {}^3\epsilon_{bcd} (E^b{}_n \alpha^n),$$

where the Ashtekar-Hansen gravi-electric field is defined by

$$\begin{aligned} E^a{}_b &:= \eta_c \eta^d \mathbb{R}^{ca}{}_{db} = \frac{8M}{(V + 1/V)^3} \left(\frac{1}{V} \partial_U + V \partial_V \right)^a \left(\frac{1}{V} dV + V dU \right)_b \\ &- \frac{8M}{(V + 1/V)^3} \tilde{E}_i^a \tilde{e}_b^i \\ &= \frac{M}{\operatorname{ch}^3 \tau} (2 \partial_\tau^a d\tau_b - \tilde{E}_i^a \tilde{e}_b^i) \quad \{V = e^\tau\}. \end{aligned}$$

*where by a slight abuse of notation with have used the same symbol α for both the Spi-translation and the translation relative to η_{ab}

Evaluating the contraction with an asymptotic translation

$$E^a{}_b \alpha^b = \frac{M}{\text{ch}^3 \tau} \left(2(\alpha^t \text{ch} \tau - (\alpha \cdot e_r) \text{sh} \tau) \partial_\tau^a - \frac{(\alpha \cdot e_\theta)}{\text{ch} \tau} \partial_\theta^a - \frac{(\alpha \cdot e_\phi)}{\text{ch} \tau \sin \theta} \partial_\phi^a \right),$$

where e_r, e_θ, e_ϕ denote the unit vectors with respect to standard spherical polar coordinates, over a $\tau = \text{const}$ section of \mathcal{D} finally gives the ADM four-momentum

$$P^{ADM} \cdot \alpha = -\frac{1}{8\pi} \int_{S_\tau^2} {}^3\epsilon_{abc} (E^a{}_m \alpha^m) = -M \alpha^t.$$

• \mathcal{I}^+ and Bondi four-momentum: Relative to coordinates which are regular in a neighborhood of \mathcal{I}^+

$$\bar{v} = \frac{1}{v} \quad \partial_v = -\bar{v}^2 \partial_{\bar{v}} \quad dv = -\frac{1}{\bar{v}^2} d\bar{v}$$

the expression for the Riemann-tensor becomes

$$\begin{aligned} \mathbf{Riemann} &= \frac{f''}{2} (\partial_u \wedge \partial_{\bar{v}}) (du \wedge d\bar{v}) \\ &+ \frac{f'}{2r} \left((\partial_u \wedge \tilde{E}_i) (du \wedge \tilde{e}^i) + (\partial_{\bar{v}} \wedge \tilde{E}_i) (d\bar{v} \wedge \tilde{e}^i) \right) \\ &+ \frac{f}{2r^2} (\tilde{E}_i \wedge \tilde{E}_j) (\tilde{e}^i \wedge \tilde{e}^j). \end{aligned}$$

Taking the limit to \mathcal{I}^+ we obtain for the rescaled Riemann (Weyl) tensor

$$\begin{aligned} \mathbb{R}^{ab}{}_{cd} &= \lim_{\bar{v} \rightarrow 0} (\Omega^{-3} R^{ab}{}_{cd}) = 16M (\partial_u \wedge \partial_{\bar{v}})^{ab} (du \wedge d\bar{v})_{cd} \\ &- 8M \left((\partial_u \wedge \tilde{E}_i)^{ab} (du \wedge \tilde{e}^i)_{cd} + (\partial_{\bar{v}} \wedge \tilde{E}_i)^{ab} (d\bar{v} \wedge \tilde{e}^i)_{cd} \right) \\ &+ 8M (\tilde{E}_i \wedge \tilde{E}_j)^{ab} (\tilde{e}^i \wedge \tilde{e}^j)_{cd}, \end{aligned}$$

and the bivector potential

$$\begin{aligned} Q^{ab} &= \lim_{\bar{v} \rightarrow 0} (\Omega^{-1} (x \wedge \alpha)^{ab}) = (\alpha^t - (\alpha \cdot e_r)) (\partial_u \wedge \partial_{\bar{v}})^{ab} \\ &+ 2u (\alpha \cdot e_\theta) (\partial_u \wedge \partial_\theta)^{ab} + 2u \frac{(\alpha \cdot e_\phi)}{\sin \theta} (\partial_u \wedge \partial_\phi)^{ab}. \end{aligned}$$

Contracting $\mathbb{R}^{ab}{}_{cd}$ with Q^{ab}

$$\begin{aligned} \frac{1}{2} \mathbb{R}^{ab}{}_{cd} Q^{cd} &= 16M (\alpha^t - (\alpha \cdot e_r)) \partial_u \wedge \partial_{\bar{v}} \\ &- 16M (\alpha \cdot e_\theta) u \partial_u \wedge \partial_\theta - 16M \frac{(\alpha \cdot e_\phi)}{\sin \theta} u \partial_u \wedge \partial_\phi, \end{aligned}$$

once again yields the “gravitational” field-strength bivector whose flux through an S_u^2 section of \mathcal{I}^+ gives the Bondi four-momentum

$$\begin{aligned} \frac{1}{4} {}^4\epsilon_{abcd} \mathbb{R}^{ab}{}_{mn} Q^{mn} &= -2M(\alpha^t - (\alpha \cdot e_r)) d^2\Omega_{cd} \\ P^{Bondi} \cdot \alpha &= \frac{1}{8\pi} \int_{S_u^2} {}^4\epsilon_{abcd} \mathbb{R}^{ab}{}_{mn} Q^{mn} \frac{1}{4} = -M\alpha^t, \end{aligned}$$

which coincides with the ADM momentum since there is no gravitational radiation in the Schwarzschild spacetime. In summary we have shown that the “Kerr-Schild compactification”, i.e. constructing the conformal boundary from the flat part of the Kerr-Schild decomposition yields the correct Bondi and ADM four-momenta.

2) *EM-pulse and AS geometry*

EM-pulse

The ultrarelativistic limit of the Coulomb field, i.e. the electromagnetic field of a point charge moving at the speed of light, is most easily constructed by boosting the corresponding four-current. Relative to the rest frame of the charge we find

$$j^a = e\delta^{(3)}(x)\partial_t^a = e\delta(Q \cdot x)\delta^{(2)}(x)P^a \quad P^a = m\partial_t^a, \quad Q^a = m\partial_z^a,$$

where P^a and Q^a represent the respectively time and spacelike vectors spanning the boost plane, which are conveniently normalized in order to allow a null limit. Denoting their common null limit by p^a the EM-pulse current becomes

$$j^a = e\delta(px)\delta^{(2)}(x)p^a$$

which in turn via Maxwell’s equations gives rise to the EM-pulse potential and field-strength

$$\begin{aligned} A &= -2e\delta(px) \log \rho \, p dx, & \rho^2 &= x^2 + y^2 \\ F &= -2e\delta(px) \frac{1}{\rho} d\rho \wedge p dx & u &= t - r \\ & & v &= t + r \\ &= -\frac{2e\delta(u(1 + \cos \theta) + v(1 - \cos \theta))}{(v - u) \sin \theta} (du(1 + \cos \theta) + dv(1 - \cos \theta) \\ &\quad + (v - u) \sin \theta d\theta) \wedge ((dv - du) \sin \theta + (v - u) \cos \theta d\theta) \\ &= -\frac{2e\delta(u(1 + \cos \theta) + v(1 - \cos \theta))}{(v - u) \sin \theta} (2du \wedge dv \sin \theta + \\ &\quad + du \wedge d\theta(1 + \cos \theta)(v - u) - dv \wedge d\theta(1 - \cos \theta)(v - u)). \end{aligned}$$

Note that field is completely concentrated on the null-hyperplane $px = -t + z = 0$ and has a singularity along the generator $\rho = 0$, which represents the support of the current of the boosted charge. Moreover, its character is completely null, i.e. $F \wedge F$ as well as $F \wedge *F$ vanish.

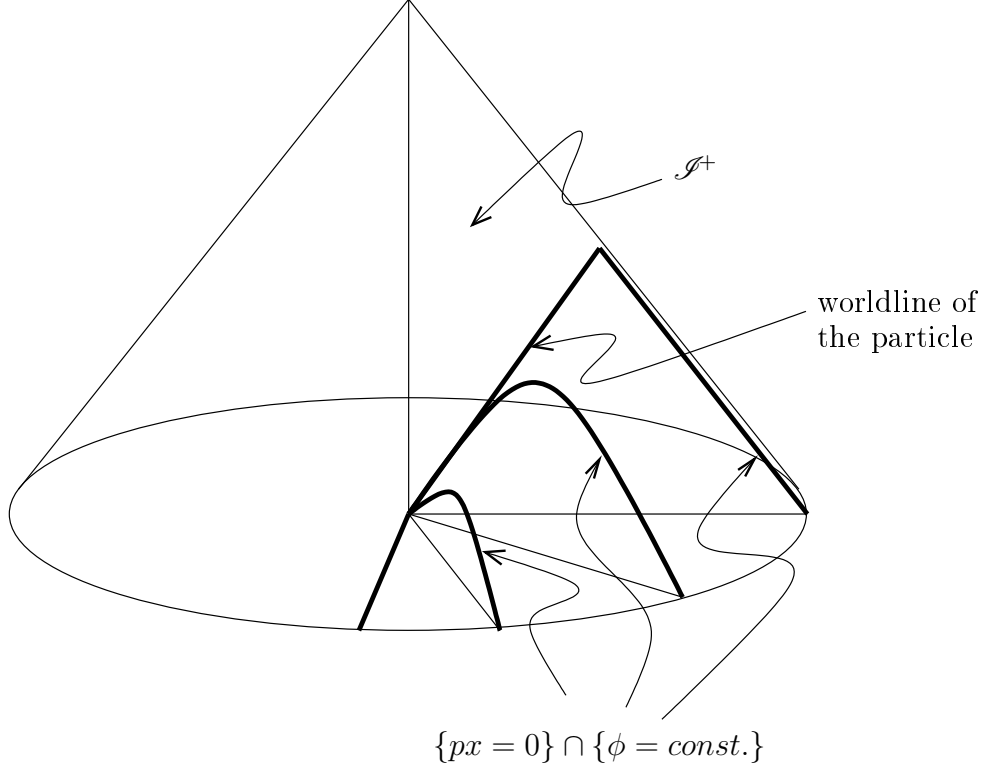


Figure 1: $\phi = \text{const}$ sections of the pulse-plane $px = 0$ in conformally compactified Minkowski space

- i^0 and ADM-charge: Proceeding analogously to the Coulomb situation the field strength becomes relative to coordinates $\bar{u} = -1/v, \bar{v} = -1/u$ regular at i^0

$$F = \frac{2e\delta(\bar{v}(1 + \cos \theta) + \bar{u}(1 - \cos \theta))}{(\bar{v} - \bar{u}) \sin \theta} (2d\bar{u} \wedge d\bar{v} \sin \theta + \\ + \frac{\bar{v}}{\bar{u}} d\bar{u} \wedge d\theta (1 + \cos \theta)(\bar{v} - \bar{u}) - \frac{\bar{u}}{\bar{v}} d\bar{v} \wedge d\theta (1 - \cos \theta)(\bar{v} - \bar{u}))$$

Taking the direction dependent of F limit to i^0 using $\bar{u} = \lambda U$ and $\bar{v} = \lambda V$ we find

$$\begin{aligned} \lim_{\lambda \rightarrow 0}(\Omega F) &= (-UV) \frac{2e\delta(U(1+\cos\theta) + V(1-\cos\theta))}{(V-U)\sin\theta} (-2dU \wedge dV \sin\theta + \\ &\quad -\frac{V}{U}(1+\cos\theta)dV \wedge d\theta(V-U) + \frac{U}{V}(1-\cos\theta)dU \wedge d\theta(V-U)), \end{aligned}$$

which is a tensor distribution on $T_{i^0}\hat{M}$. the field at infinity is completely determined by its restriction to $\mathcal{D} : UV = -1$

$$\begin{aligned} \mathbb{F} &:= \lim_{\lambda \rightarrow 0}(\Omega F)|_{\mathcal{D}} = \frac{2e\delta((1/V)(1+\cos\theta) - V(1-\cos\theta))}{(V+1/V)\sin\theta} (-2dU \wedge dV \sin\theta + \\ &\quad + V^2(1+\cos\theta)dV \wedge d\theta(V+1/V) - \frac{1}{V^2}(1-\cos\theta)dU \wedge d\theta(V+1/V)). \end{aligned}$$

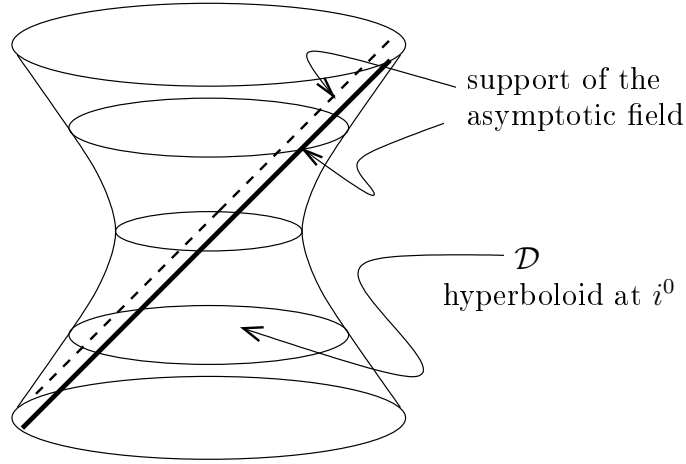


Figure 2: Support of the field strength on the set \mathcal{D} of spacelike directions

Integrating the flux 2-form over an S^2 section of \mathcal{D} gives the ADM charge

$$Q^{ADM} = -\frac{1}{4\pi} \int_{S^2} \frac{1}{2} \epsilon_{abcd} \mathbb{F}^{ab} = \frac{1}{4\pi} \int_{S^2} {}^3\epsilon_{abc} E^a,$$

where the electric field in the last equation is obtained from

$$\begin{aligned}
E &= \eta \rfloor \mathbb{F} = 2e \frac{\delta((1/V)(1 + \cos \theta) - V(1 - \cos \theta))}{(V + 1/V) \sin \theta} \\
&\quad (2 \sin \theta ((1/V)dV + VdU) - V(1 + \cos \theta)(V + 1/V)d\theta + \\
&\quad + (1/V)(1 - \cos \theta)(V + 1/V)d\theta) = \\
&= 2e \frac{\delta(\text{th } \tau - \cos \theta)}{\text{ch}^2 \tau} (d\tau + \text{ch } \tau d\theta) = 2ed\Theta(\text{th } \tau - \cos \theta) \quad \{V = e^\tau\}.
\end{aligned}$$

The last expression explicitly exhibits the exactness of E , which automatically entails its closedness. Let us now show that it is also co-closed, i.e. $D_a E^a = 0$. Using the volume form together with a null triad on \mathcal{D}

$$\begin{aligned}
\omega &= p \wedge \bar{p} \wedge \text{ch } \tau \sin \theta d\phi, \\
p &= d\tau + \text{ch } \tau d\theta \quad \bar{p} = d\tau - \text{ch } \tau d\theta, \quad p^2 = \bar{p}^2 = 0, \quad p \cdot \bar{p} = -1,
\end{aligned}$$

we find

$$\begin{aligned}
*E = E \rfloor \omega &= \frac{2e\delta}{\cosh^2 \tau} p \wedge \cosh \tau \sin \theta d\phi = \delta \left(\frac{d\tau}{\cosh^2 \tau} + \sin \theta d\theta \right) \wedge d\phi \\
&= 2ed\Theta(\tanh \tau - \cos \theta) \wedge d\phi \quad \Rightarrow \quad d * E = 0.
\end{aligned}$$

Therefore its flux-integral over any (topologically S^2) section of \mathcal{D} has the same value

$$Q^{ADM} = \frac{1}{4\pi} \int_{S_\tau^2} \epsilon_{abc} E^a = e.$$

• \mathcal{J}^+ and Bondi-charge: In a neighborhood of \mathcal{J}^+ we find the following expression for the electromagnetic field-strength

$$\begin{aligned}
F &= 2e\delta(u\bar{v}(1 + \cos \theta) + (1 - \cos \theta)) \\
&\quad \left(\frac{2}{1 - u\bar{v}} du \wedge d\bar{v} - \bar{v} \frac{1 + \cos \theta}{\sin \theta} du \wedge d\theta - \frac{1}{\bar{v}} \frac{1 - \cos \theta}{\sin \theta} d\bar{v} \wedge d\theta \right) \\
&\quad \bar{v} = 1/v.
\end{aligned}$$

Taking into account that in the limit $\bar{v} \rightarrow 0$ the support of the δ -function tends to $\theta = 0$, the north pole of S^2 , we find by using[†]

$$\begin{aligned}
\lim_{\bar{v} \rightarrow 0} (\delta(2u\bar{v} + \frac{\theta^2}{2}), \tilde{\varphi}) &= \theta(-u) \int d\phi \varphi(2\sqrt{-u\bar{v}}e_\theta) = \\
&= 2\pi(\theta(-u)\delta_N^{(2)}(x), \tilde{\varphi}), \\
\lim_{\bar{v} \rightarrow 0} (\bar{v}\delta(2u\bar{v} + \frac{\theta^2}{2})\frac{2}{\theta}e_\theta^i, \tilde{\varphi}) &= \theta(-u)\bar{v} \int d\phi \frac{e_\theta^i}{\sqrt{-u\bar{v}}} \varphi(2\sqrt{-u\bar{v}}e_\theta) = 0, \\
\lim_{\bar{v} \rightarrow 0} (\frac{1}{\bar{v}}\delta(2u\bar{v} + \frac{\theta^2}{2})\frac{\theta}{2}e_\theta^i, \tilde{\varphi}) &= \theta(-u)\frac{1}{\bar{v}} \int d\phi \sqrt{-u\bar{v}}e_\theta^i \varphi(2\sqrt{-u\bar{v}}e_\theta) = \\
&= 2\pi(u\theta(-u)\partial^i \delta_N^{(2)}(x), \tilde{\varphi}),
\end{aligned}$$

where $\delta_N^{(2)}(x)$ denotes the δ -function concentrated on the north pole ($\theta = 0$) of the 2-sphere and e_θ^i the “radial” unit-vector, that the asymptotic field-strength is given by

$$\mathbb{F} = 4\pi e(2\theta(-u)\delta_N^{(2)}(x)du + u\theta(-u)dx^i\partial_i\delta_N^{(2)}(x)) \wedge d\bar{v}.$$

Integration of the flux 2-form over an S^2 -section of \mathcal{I}^+ yields the Bondi-charge

$$Q^{Bondi}(u) = \frac{1}{4\pi} \int_{S^2} *\mathbb{F} = e\theta(-u) \int_{S_u^2} \delta_N^{(2)}(x)d^2\Omega = e\theta(-u).$$

This shows that the charge reaches null infinity at the retarded time $u = 0$. The last integration depends only of the “location” of the S^2 relative to the point $u = \theta = 0$ on \mathcal{I}^+ as can be seen from

$$d * \mathbb{F}|_{\mathcal{I}^+} = 4\pi e\delta(u)\delta_N^{(2)}(x)du \wedge d^2\Omega. \quad (2)$$

The right hand side is actually nothing but the limit of the current to \mathcal{I}^+ as can be seen from

[†]Note that the support of the δ -function tends to the north-pole, where spherical polar coordinates become singular. Nevertheless using regular (stereographic) coordinates leads to the same result.

$$\begin{aligned}
j &= e\delta(px)\delta^{(2)}(x)(pdx) = 4e\delta(u(1+\cos\theta) + v(1-\cos\theta))\frac{\delta(\sin\theta(v-u))}{2\pi\sin\theta(v-u)} \\
&\quad (du(1+\cos\theta) + dv(1-\cos\theta) + (v-u)\sin\theta d\theta) = \{\bar{v} = 1/v\} = \\
&\quad = 4e\bar{v}^3\delta(\bar{v}u(1+\cos\theta) + (1-\cos\theta))\frac{\delta(\sin\theta(1-u\bar{v}))}{2\pi\sin\theta(1-u\bar{v})} \\
&\quad (du(1+\cos\theta) - \frac{d\bar{v}}{\bar{v}^2}(1-\cos\theta) + \bar{v}(1-u\bar{v})\sin\theta d\theta) \\
\mathbb{J} &= \lim_{\bar{v}\rightarrow 0}(\Omega^{-2}j) = e\delta(u)\delta^{(2)}(x)du,
\end{aligned}$$

which is nothing but the dual of (2). Finally, let us consider the limit of the vector potential

$$\begin{aligned}
A &= -2e\delta(px)\log\rho pdx = -2e\delta(u(1+\cos\theta) + v(1-\cos\theta)) \\
&\log\left(\frac{v-u}{2}\sin\theta\right)(du(1+\cos\theta) + dv(1-\cos\theta) + (v-u)\sin\theta d\theta) = \\
&\quad -2e\delta(2u\bar{v} + \frac{\theta^2}{2})(\log\frac{1-u\bar{v}}{2} + \log\sqrt{-4u\bar{v}} - \log\bar{v}) \\
&\quad (2\bar{v}du + 4ud\bar{v} + (1-u\bar{v})\sqrt{-4u\bar{v}}d\theta) = \\
&\quad -2e\delta(2u\bar{v} + \frac{\theta^2}{2})(\log((1-u\bar{v})\sqrt{-u}) - \frac{1}{2}\log\bar{v}) \\
&\quad (2\bar{v}du + 4ud\bar{v} + (1-u\bar{v})\sqrt{-4u\bar{v}}d\theta) \quad (3)
\end{aligned}$$

As can be seen from (3) the presence of the $\log\bar{v}$ spoil the existence of the limit. However, if one takes a closer look at these terms one notices that they are actually pure gauge, i.e. of the form $\log\bar{v}\delta(px)pdx$. They may therefore be removed by a gauge transformation, which implies that

$$\lim_{\bar{v}\rightarrow 0}(A - e\log\bar{v}\delta(px)pdx) = -4eu\theta(-u)\log(-u)\delta_N^{(2)}(x)d\bar{v}.$$

Since the corresponding (gauge-invariant) field-strength F exists this result might actually have been anticipated.

AS-geometry

There is a striking similarity between the electromagnetic field for an ultra-relativistic charge and the gravitational field of an ultrarelativistic black hole. The essential difference is that spacetime is no longer Minkowskian and it is not immediately clear how to construct a suitable boundary. At this point it is important to recall that metric (1) being an impulsive pp-wave lies in the

intersection of two larger families of geometries: the Kerr-Schild spacetimes, which contain pp-waves as subclass on the one hand and impulsive gravitational waves on the other hand, whose curvature is concentrated on a null hypersurface. When constructing the asymptotic structure we make use of both of these properties: The Kerr-Schild form implies that the manifold is equipped with two metrical structures: a flat one η_{ab} and the physical one g_{ab} ; that the curvature is concentrated on a submanifold of co-dimension one implies that space is flat almost everywhere. To investigate the asymptotic structure we compactify the spacetime. We assume that the boundary of the compactified manifold has the same topology as compactified Minkowski space i.e. $\mathbb{R} \times S^2 \cup \text{point}$. Since spacetime is flat almost everywhere we apply the usual conformal rescaling of flat spacetime to g_{ab} and show that it has a well-defined (distributional) meaning on the boundary. The singular nullplane where the curvature is concentrated cuts \mathcal{S}^+ along a single generator up to the point $u = 0$. Thus the conformally transformed physical metric will be smooth everywhere on \mathcal{S}^+ except for this half generator. As we shall show, on the half-generator the metric can be given distributional meaning; actually g_{ab} is continuous. Although the conformal connection is ill defined all the quantities needed to define Bondi four momentum do exist. Going out to spatial infinity i^0 the situation is similar: spacelike directions lying in the singular plane trace out a topological cylinder, i.e. $\mathbb{R} \times S^1$, on the hyperboloid \mathcal{D} in the tangent space at i^0 ($\theta = \pi/2, 0 \leq \phi \leq 2\pi$). Once again it can be shown that these contributions of the non-flat part of the metric vanish in a distributional sense and that the ADM four-momentum is well defined.

As already mentioned the metric (1) is flat almost everywhere. Actually a stronger statement, namely that (1) is asymptotic to η_{ab} can be proven. To this end let us consider a conformal factor Ω that gives rise to the standard (Penrose) compactification of Minkowski space. Multiplication of (1) with Ω^2 gives

$$\begin{aligned} \tilde{g}_{ab} &:= \Omega^2 g_{ab} = \tilde{\eta}_{ab} + f(\Omega p)_a (\Omega p)_b = \\ &= \tilde{\eta}_{ab} + \delta(\Omega \cdot (px)) \Omega (\log(\Omega \rho) - \log \Omega) (\Omega p)_a (\Omega p)_b, \end{aligned}$$

where the negative homogeneity of δ has been used in the last equality. Since Ωp_a , $\Omega \cdot (px)$ and $\Omega \cdot \rho$ all tend to well-defined (finite) quantities in the limit $\Omega \rightarrow 0$ (at the support of δ) it is easy to see that remaining (infinite) $\log \Omega$ cannot outrun Ω . Thus we find

$$\lim_{\Omega \rightarrow 0} \tilde{g}_{ab} = \lim_{\Omega \rightarrow 0} \tilde{\eta}_{ab},$$

which explicitly shows that in a suitable framework (i.e. distribution theory) the AS-geometry tends to the flat metric. Although the metric itself is

well-defined and tends to Minkowski at the boundary the connection is not differentiable. This is most easily exhibited by taking the \bar{v} -derivative of

$$\Omega^2 ds^2 = dud\bar{v} + \frac{(1-u\bar{v})^2}{4}d\Omega^2 + \delta(u\bar{v}(1+\cos\theta) + (1-\cos\theta)) \\ (\bar{v}du(1+\cos\theta) - \frac{d\bar{v}}{\bar{v}}(1-\cos\theta - \sin\theta(1-u\bar{v})d\theta)^2 \frac{\bar{v}}{2}(\log \frac{1-u\bar{v}}{2} \sin\theta - \log \bar{v})).$$

Obviously the last factor contains a product of the form $\bar{v} \log \bar{v}$ which is not \bar{v} -differentiable at $\bar{v} = 0$. However this term is precisely of the same form as the singular term in the gauge potential for the electromagnetic shock wave. It is thus reasonable to assume that it may be eliminated in the same manner by using a singular coordinate transformation. Although we will not go into the details of this construction, this assumption is a posteriori justified since the gauge-invariant quantities like the Riemann and Weyl-tensors have well-defined limits as will be shown in the following.

The Riemann tensor for an arbitrary pp-wave is given by

$$R^{ab}{}_{cd} = -2p^{[a}\partial^b]p_{[c}\partial_{d]}f.$$

Proceeding in an analogous manner as for the Schwarzschild geometry and taking into account that the wave profile for the AS-geometry is given by $f = -8\mu \log \rho \delta(px)$, we obtain

$$\begin{aligned} \mathbf{Riemann} = & \frac{16\mu\delta(u(1+\cos\theta) + v(1-\cos\theta))}{(v-u)^2 \sin^2\theta} [(\sin\theta(\partial_u \wedge \partial_v) - \\ & - \frac{1-\cos\theta}{v-u}(\partial_u \wedge \partial_\theta) + \frac{1+\cos\theta}{v-u}(\partial_v \wedge \partial_\theta))(2\sin\theta(du \wedge dv) + \\ & + (1+\cos\theta)(v-u)(du \wedge d\theta) - (1-\cos\theta)(v-u)(dv \wedge d\theta)) - \\ & - ((1-\cos\theta)\partial_u + (1+\cos\theta)\partial_v - \frac{2\sin\theta}{(v-u)}\partial_\theta) \wedge \partial_\phi \\ & ((1+\cos\theta)du + (1-\cos\theta)dv + \sin\theta(v-u)d\theta) \wedge d\phi]. \end{aligned}$$

where u and v denote the standard (spherical) retarded and advanced coordinates.

- i^0 and ADM-momentum: For regular coordinates at spatial infinity $\bar{u} =$

$-1/v, \bar{v} = -1/u$ the expression for the Riemann tensor becomes

$$\begin{aligned} \mathbf{Riemann} = & -(\bar{u}\bar{v})^3 \frac{16\mu\delta(\bar{u}(1+\cos\theta) + \bar{v}(1-\cos\theta))}{(\bar{v}-\bar{u})^2 \sin^2\theta} [(\sin\theta(\partial_{\bar{u}} \wedge \partial_{\bar{v}}) + \\ & -\frac{\bar{v}}{\bar{u}} \frac{1-\cos\theta}{\bar{v}-\bar{u}}(\partial_{\bar{v}} \wedge \partial_{\theta}) + \frac{\bar{u}}{\bar{v}} \frac{1+\cos\theta}{\bar{v}-\bar{u}}(\partial_{\bar{u}} \wedge \partial_{\theta}))(2\sin\theta(d\bar{u} \wedge d\bar{v}) + \\ & +(1+\cos\theta)\frac{\bar{u}}{\bar{v}}(\bar{v}-\bar{u})(d\bar{v} \wedge d\theta) - (1-\cos\theta)\frac{\bar{v}}{\bar{u}}(\bar{v}-\bar{u})(d\bar{u} \wedge d\theta)) - \\ & -((1-\cos\theta)\frac{\bar{v}}{\bar{u}}\partial_{\bar{v}} + (1+\cos\theta)\frac{\bar{u}}{\bar{v}}\partial_{\bar{u}} + \frac{2\sin\theta}{(\bar{v}-\bar{u})}\partial_{\theta}) \wedge \partial_{\phi} \\ & (\frac{\bar{u}}{\bar{v}}(1+\cos\theta)d\bar{v} + \frac{\bar{v}}{\bar{u}}(1-\cos\theta)d\bar{u} - \sin\theta(\bar{v}-\bar{u})d\theta) \wedge d\phi]. \end{aligned}$$

Taking the direction dependent limit using $\bar{u} = \lambda U, \bar{v} = \lambda V$ and $\lambda \rightarrow 0$ we find

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \Omega^{-3/2} \mathbf{Riemann} = & (-UV)^{3/2} \frac{16\mu\delta(U(1+\cos\theta) + V(1-\cos\theta))}{(V-U)^2 \sin^2\theta} [(\sin\theta \\ & (\partial_U \wedge \partial_V) - \frac{V}{U} \frac{1-\cos\theta}{V-U}(\partial_V \wedge \partial_{\theta}) + \frac{U}{V} \frac{1+\cos\theta}{V-U}(\partial_U \wedge \partial_{\theta}))(2\sin\theta(dU \wedge dV) + \\ & +(1+\cos\theta)\frac{U}{V}(V-U)(dV \wedge d\theta) - (1-\cos\theta)\frac{V}{U}(V-U)(dU \wedge d\theta)) - \\ & -((1-\cos\theta)\frac{V}{U}\partial_V + (1+\cos\theta)\frac{U}{V}\partial_U + \frac{2\sin\theta}{V-U}\partial_{\theta}) \wedge \partial_{\phi} \\ & (\frac{U}{V}(1+\cos\theta)dV + \frac{V}{U}(1-\cos\theta)dU - \sin\theta(V-U)d\theta) \wedge d\phi]. \end{aligned}$$

Due to the (infinite) rescaling, i.e. $\lambda \rightarrow 0$, only the projective information is physically relevant, which is conveniently encoded in the restriction to the hyperboloid $\mathcal{D} : UV = -1$

$$\begin{aligned} \mathbb{R}^{ab}{}_{cd} & := \lim_{\lambda \rightarrow 0} \Omega^{-3/2} R^{ab}{}_{cd}|_{\mathcal{D}} = \frac{16\mu\delta((1/V)(1+\cos\theta) - V(1-\cos\theta))}{(V+1/V)^2 \sin^2\theta} [(\sin\theta + \\ & (\partial_U \wedge \partial_V)^{ab} + V^2 \frac{1-\cos\theta}{V+1/V}(\partial_V \wedge \partial_{\theta})^{ab} - \frac{1}{V^2} \frac{1+\cos\theta}{V+1/V}(\partial_U \wedge \partial_{\theta})^{ab}) \\ & (2\sin\theta(dU \wedge dV)_{cd} + -(1+\cos\theta)\frac{1}{V^2}(V+1/V)(dV \wedge d\theta)_{cd} + \\ & +(1-\cos\theta)V^2(V+1/V)(dU \wedge d\theta)_{cd} - (((1-\cos\theta)V^2\partial_V + \\ & (1+\cos\theta)\frac{1}{V^2}\partial_U - \frac{2\sin\theta}{V+1/V}\partial_{\theta}) \wedge \partial_{\phi})^{ab} ((\frac{1}{V^2}(1+\cos\theta)dV + \\ & V^2(1-\cos\theta)dU + \sin\theta(V+1/V)d\theta) \wedge d\phi)_{cd}]. \end{aligned}$$

As for the Schwarzschild solution the dual of the contraction of \mathbb{R}^{ab}_{cd} with the bivector potential Q^{ab} gives the flux 2-form. Its corresponding integral over an S^2 -section of \mathcal{D} can be expressed in terms of the Ashtekar-Hansen gravi-electric field $E^a_b = \eta_c \eta^d \mathbb{R}^{ab}_{cd}$

$$\begin{aligned} E^a_b = & \frac{16\mu\delta((1/V)(1+\cos\theta) - V(1-\cos\theta))}{(V+1/V)^2 \sin^2\theta} (\sin\theta(V\partial_V + \frac{1}{V}\partial_U)^a - \\ & (1 - \frac{V-1/V}{V+1/V}\cos\theta)\partial_\theta^a)(\sin\theta(\frac{1}{V}dV + VdU)_b + \\ & (1 - \frac{V-1/V}{V+1/V}\cos\theta)\frac{(V+1/V)^2}{2}d\theta_b) = \\ & 4\mu\frac{\delta(\text{th}\tau - \cos\theta)}{\text{ch}^3\tau}(\partial_\tau - \frac{1}{\text{ch}\tau}\partial_\theta)^a(d\tau + \text{ch}\tau d\theta)_b. \end{aligned}$$

The ADM-momentum therefore becomes

$$\begin{aligned} P^{ADM} \cdot \alpha = & -\frac{1}{8\pi} \int_{S^2_\tau} {}^3\epsilon_{abc} E^c_n \alpha^n = -\frac{\mu}{2\pi} \int d^2\Omega \frac{\delta(\text{th}\tau - \cos\theta)}{\text{ch}\tau} \\ & (\alpha^t \text{ch}\tau - \text{sh}\tau(\alpha \cdot e_r) + (\alpha \cdot e_\theta)) = -\mu(\alpha^t - \alpha^z) = \mu(p \cdot \alpha), \end{aligned}$$

where the integral in the penultimate equality is a formal expression denoting the evaluation of a (compactly-supported) distribution on the unit test-function. The calculation is actually independent of the S^2 section of \mathcal{D} as can be seen from the fact that E^a_b is conserved, i.e. $D_a E^a_b = 0$. Thus the ADM-momentum is null and proportional to μ .

• \mathcal{J}^+ and Bondi-momentum: Unlike the Schwarzschild situation the impulsive gravitational wave is generated by a null source. Therefore we will first investigate the behaviour of the latter on the conformal boundary. This is important, since a failure in the fall-off of the matter distribution would spoil asymptotic flatness.

$$\begin{aligned} T_{ab} = & \mu\delta(px)\delta^{(2)}(x)(pdx)_{ab}^2 = \\ & 2\delta(u(1+\cos\theta) + v(1-\cos\theta))\frac{\delta((v-u)\sin\theta)}{2\pi(v-u)\sin\theta} \\ & (du(1+\cos\theta) + dv(1-\cos\theta) + (v-u)d\theta)_{ab}^2 = \bar{v} = \frac{1}{v} \\ & 2\bar{v}^3\mu\delta(u\bar{v}(1+\cos\theta) + (1-\cos\theta))\frac{\delta((1-u\bar{v})\sin\theta)}{2\pi(1-u\bar{v})\sin\theta} \\ & (du(1+\cos\theta) - \frac{d\bar{v}}{\bar{v}}(1-\cos\theta) + \frac{1}{\bar{v}}(1-u\bar{v})\sin\theta d\theta)_{ab}^2 \end{aligned}$$

Taking the limit $\bar{v} \rightarrow 0$ we find

$$\lim_{\bar{v} \rightarrow 0} \Omega^{-2} T_{ab} = \mu \delta(u) \delta_N^{(2)}(x) (du^2)_{ab}.$$

This result implies that the energy momentum vanishes at the conformal boundary and therefore does not compromise asymptotic flatness. Moreover, the scaling behavior is complete agreement with [7]. Having settled this issue, we begin the calculation of the Bondi-momentum by transforming the expression for the Riemann-tensor to coordinates regular in a neighborhood of \mathcal{I}^+ by setting $\bar{v} = 1/v$

$$\begin{aligned} \mathbf{Riemann} = & 16\mu \frac{\bar{v}^3 \delta(\frac{\theta^2}{2} + 2u\bar{v})}{\theta^2} [2\theta^2 (\partial_u \wedge \partial_{\bar{v}})(du \wedge d\bar{v}) \\ & - 2\theta\bar{v} (\partial_u \wedge \partial_{\bar{v}})(du \wedge d\theta) - \frac{\theta^3}{2\bar{v}} (\partial_u \wedge \partial_{\bar{v}})(d\bar{v} \wedge d\theta) + \frac{\theta^3}{\bar{v}} (\partial_u \wedge \partial_{\theta})(du \wedge d\bar{v}) \\ & - \theta^2 (\partial_u \wedge \partial_{\theta})(du \wedge d\theta) - \frac{\theta^4}{4\bar{v}^2} (\partial_u \wedge \partial_{\theta})(d\bar{v} \wedge d\theta) + 4\theta\bar{v} (\partial_{\bar{v}} \wedge \partial_{\theta})(du \wedge d\bar{v}) \\ & - 4\bar{v}^2 (\partial_{\bar{v}} \wedge \partial_{\theta})(du \wedge d\theta) - \theta^2 (\partial_{\bar{v}} \wedge \partial_{\theta})(d\bar{v} \wedge d\theta) - \theta^2 (\partial_u \wedge \partial_{\phi})(du \wedge d\phi) \\ & + \frac{\theta^4}{4\bar{v}^2} (\partial_u \wedge \partial_{\phi})(d\bar{v} \wedge d\phi) - \frac{\theta^3}{2\bar{v}} (\partial_u \wedge \partial_{\phi})(d\theta \wedge d\phi) + 4\bar{v}^2 (\partial_{\bar{v}} \wedge \partial_{\phi})(du \wedge d\phi) \\ & - \theta^2 (\partial_{\bar{v}} \wedge \partial_{\phi})(d\bar{v} \wedge d\phi) + 2\theta\bar{v} (\partial_{\bar{v}} \wedge \partial_{\phi})(d\theta \wedge d\phi) + 4\theta\bar{v} (\partial_{\theta} \wedge \partial_{\phi})(du \wedge d\phi) + \\ & - \frac{\theta^3}{\bar{v}} (\partial_{\theta} \wedge \partial_{\phi})(d\bar{v} \wedge d\phi) + 2\theta^2 (\partial_{\theta} \wedge \partial_{\phi})(d\theta \wedge d\phi)] \end{aligned} \quad (4)$$

In the above equation we have made use of the fact that the support of the delta-function concentrates around the north-pole of the S^2 in the limit $\bar{v} \rightarrow 0$.

$$\begin{aligned} \mathbb{R}^{ab}_{cd} = & \lim_{\bar{v} \rightarrow 0} (\Omega^{-3} R^{ab}_{cd}) = 32\pi\theta(-u) [2\delta_N^{(2)}(x) (\partial_u \wedge \partial_{\bar{v}})^{ab} (du \wedge d\bar{v})_{cd} + \\ & + u\partial_i \delta_N^{(2)}(x) (\partial_u \wedge \partial_{\bar{v}})^{ab} (d\bar{v} \wedge dx^i)_{cd} - 2u\partial^i \delta_N^{(2)}(x) (\partial_u \wedge \partial_i)^{ab} (du \wedge d\bar{v})_{cd} + \\ & + \delta_N^{(2)}(x) (\partial_u \wedge \partial_i)^{ab} (du \wedge dx^i)_{cd} + \delta_N^{(2)}(x) (\partial_{\bar{v}} \wedge \partial_i)^{ab} (d\bar{v} \wedge dx^i)_{cd} + \\ & + u\partial_j \delta_N^{(2)}(x) (\partial_u \wedge \partial_i)^{ab} (dx^j \wedge dx^i)_{cd} + 2u\partial^i \delta_N^{(2)}(x) (\partial_i \wedge \partial_j)^{ab} (d\bar{v} \wedge dx^j)_{cd} - \\ & - \delta_N^{(2)}(x) (\partial_i \wedge \partial_j)^{ab} (dx^i \wedge dx^j)_{cd} + u^2 (\partial^i \partial_j - \frac{1}{2} \delta_j^i \partial^2) \delta_N^{(2)}(x) \\ & (\partial_u \wedge \partial_i)^{ab} (d\bar{v} \wedge dx^j)_{cd}] \end{aligned} \quad (5)$$

where x^i refer to the standard two-dimensional Cartesian coordinates associated with circular polar coordinates. Let us emphasize that in the limit expression (5) not all the terms of (4) have finite limits by themselves but fortunately these infinities cancel out in the final result. The relevant integrals

for deriving these identities arise from calculations similar to

$$\begin{aligned}
\lim_{\bar{v} \rightarrow 0} \left(\frac{\theta^2}{\bar{v}^2} \delta \left(\frac{\theta^2}{2} + 2u\bar{v} \right) (e_\theta^i e_\theta^j - e_\phi^i e_\phi^j), \tilde{\varphi} \right) &= \\
= \theta(-u) \int d\phi \frac{-4u\bar{v}}{\bar{v}^2} (e_\theta^i e_\theta^j - e_\phi^i e_\phi^j) \varphi(\sqrt{-4u\bar{v}} e_\theta) &= \\
= 8u^2 \theta(-u) \int d\phi (e_\theta^i e_\theta^j - e_\phi^i e_\phi^j) e_\theta^k e_\theta^l \partial_k \partial_l \varphi(0) &= \\
= 4u^2 \theta(-u) \pi (-\delta^{ij} \partial^2 \varphi(0) + 2\partial^i \partial^j \varphi(0)). &
\end{aligned}$$

This result shows that, as in the case of the energy momentum tensor the (rescaled) Weyl-tensor has a well-defined distributional limit which is completely concentrated on the generator of \mathcal{I}^+ corresponding to the north pole of the 2-sphere for negative retarded times. Taking the integral of the flux 2-form $\frac{1}{2} {}^4\epsilon_{abcd}(\mathbb{R}^{ab} {}_{cd} Q^{cd} \frac{1}{2})$ finally yields the Bondi four-momentum

$$\begin{aligned}
P^{Bondi} \cdot \alpha &= \frac{1}{8\pi} \int_{S_u^2} \frac{1}{2} {}^4\epsilon_{abcd}(\mathbb{R}^{ab} {}_{cd} Q^{cd} \frac{1}{2}) = \\
&= -\frac{32\mu\pi}{8\pi} \theta(-u) \int d^2\Omega \frac{1}{4} \delta_N^{(2)}(x) (\alpha^t - (a \cdot e_r)) = \mu \theta(-u) (p \cdot \alpha).
\end{aligned}$$

This confirms our previously obtained result that the Bondi-momentum is null and equal to ADM up to the instant of retarded time u , where both particle and wave reach infinity and drops to zero. All energy escapes to \mathcal{I}^+ which is flat thereafter.

Conclusion

In this paper we have taken some effort to put on a stricter mathematical footing a result which on physical grounds is intuitively expected: The boosted Schwarzschild black hole looks more and more like a gravitational wave as the boost parameter tends to the velocity of light. Therefore, if in the limit process the energy is kept constant, the resulting gravitational wave should have a lightlike ADM and Bondi four-momentum. However, since there are theorems which exclude the existence of light-like total four-momenta in general relativity [8] on the basis of regularity assumptions for spacetime, the limiting metric has to violate these. In fact the curvature is of distributional nature concentrated on a null hypersurface. Nevertheless, we have shown that it is possible to construct a conformal boundary for this metric in a distributional sense. This implies that some quantities are only defined over the space of testfunctions (forms) at spatial and null infinity.

Since the integrand for the Bondi-momentum is proportional to $\theta(-u)$, the Bondi-news square would behave like $\delta(u)$. Therefore the news-function itself is not defined within the framework of classical distribution theory.

Bicak and Schmidt have analyzed boost symmetric spacetimes [9]. Specific examples represent the field of two uniformly accelerated particles. These spacetimes have a regular \mathcal{S} except at points where the particles reach infinity. Similarly, the trajectory of the ultrarelativistic “particle” meets \mathcal{S} in a point. However, we have shown that the associated stress-energy tensor vanishes at the conformal boundary. The reason for this is that the mass parameter of the Schwarzschild black hole tends to zero in the ultra-relativistic limit. Therefore, in our case there are no punctures in \mathcal{S} , which has been shown using distributional techniques. However, it should be kept in mind that the framework of distributions does in general not allow to assign point values.

Among the class of pp-waves the AS metric can be considered to be on the “edge” of asymptotic flatness. It is just the right combination $\delta \cdot \log$ terms in the wave profile which renders a finite non-zero four-momenta: Widening the impulsive profile to would make the four-momenta undefined, while replacing the log-term by a (stronger) falloff, gives a zero total momentum (see [10])

Although our construction refers to a specific spacetime it seems reasonable to believe that distributional techniques may be applicable to asymptopia in a more general context. Mathematically this due to the fact that in the asymptotically flat situation gravity becomes weak “far away” from the sources and may be treated in a linear approximation. However, classical distribution theory is tailor-made to deal with non-smooth, linear situations, thereby transferring the burden of differentiability from the boundary manifold to the space of (generalized) functions thereon.

Acknowledgments: One of us (PCA) thanks A. Ashtekar and B. Schmidt for clarifying discussions during a program at the ITP of the UCSB. This program was supported by the NSF grant no. PHY 94-07194. Partial support by the Fundacion Federico is also acknowledged.

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